GR Reference Sheet:
Einstein Maxwell Theory

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Equations of Motion

Einstein gravity coupled to the Maxwell electromagnetic field with SI units and MTW sign conventions:

$$S[g_{\mu\nu}, A_\sigma] = \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}R - \frac{1}{4}\sqrt{\frac{\epsilon_0}{\mu_0}}\int d^4x\sqrt{-g}F^{\mu\nu}F_{\mu\nu} + \frac{1}{c}\int d^4x\sqrt{-g}A_\mu J^\mu(g,\psi) + S_\psi[g,\psi] \tag{1}$$

where G is Newton?s constant. The speed of light is c, the permittivity of free space is ϵ_0, and the permeability of free space is μ_0; these are related by $\epsilon_0\mu_0 c^2 = 1$. The field strength tensor is $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and $J^\sigma(g,\psi)$ is an "external" charge current that depends on the metric $g_{\mu\nu}$ and some matter fields ψ^I. (We assume non-derivative coupling; that is, J^μ doesn't depend on derivatives of the metric.) The action for the fields ψ^I is denoted $S_\psi[g,\psi]$.

The variation of the action gives

$$\frac{\delta S}{\delta g_{\mu\nu}} = \sqrt{-g}\left\{-\frac{c^3}{16\pi G}G^{\mu\nu} + \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\left(F^{\mu\alpha}F^{\nu\beta}g_{\alpha\beta} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g^{\mu\nu}\right) + \frac{1}{2c}A_\sigma J^\sigma g^{\mu\nu} + \frac{1}{c}A_\sigma\frac{\partial J^\sigma}{\partial g_{\mu\nu}}\right\} + \frac{\delta S_\psi}{\delta g_{\mu\nu}} \tag{2}$$

$$\frac{\delta S}{\delta A_\sigma} = \sqrt{-g}\left\{\sqrt{\frac{\epsilon_0}{\mu_0}}\nabla_\mu F^{\mu\sigma} + \frac{1}{c}J^\sigma\right\} \tag{3}$$

$$\frac{\delta S}{\delta \psi^I} = \sqrt{-g}\left\{\frac{1}{c}A_\sigma\frac{\partial J^\sigma}{\partial\psi^I}\right\} + \frac{\delta S_\psi}{\delta\psi^I} \tag{4}$$

so the equations of motion are

$$G^{\mu\nu} = \frac{8\pi G}{c^4}T^{\mu\nu} \tag{5}$$

$$\nabla_\mu F^{\mu\sigma} = -\mu_0 J^\sigma \tag{6}$$

$$\frac{\delta S_\psi}{\delta\psi^I} = -\sqrt{-g}\left\{\frac{1}{c}A_\sigma\frac{\partial J^\sigma}{\partial\psi^I}\right\} \tag{7}$$

with the stress–energy–momentum (SEM) tensor

$$T^{\mu\nu} \equiv \frac{2c}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g_{\mu\nu}} = \frac{1}{\mu_0}\left(F^{\mu\alpha}F^{\nu\beta}g_{\alpha\beta} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g^{\mu\nu}\right) + A_\sigma J^\sigma g^{\mu\nu} + 2A_\sigma\frac{\partial J^\sigma}{\partial g_{\mu\nu}} + \frac{2c}{\sqrt{-g}}\frac{\delta S_\psi}{\delta g_{\mu\nu}} \tag{8}$$

The SEM tensor is conserved, $\nabla_\nu T^{\mu\nu} = 0$, when the equations of motion hold. We can compute directly the covariant divergence of the EM part of the SEM tensor:

$$\nabla_\nu T^{\mu\nu}_{EM} = \frac{1}{\mu_0}\left(\nabla_\nu F^{\mu\alpha}F^{\nu\beta}g_{\alpha\beta} + F^{\mu\alpha}\nabla_\nu F^{\nu\beta}g_{\alpha\beta} - \frac{1}{2}F^{\alpha\beta}\nabla^\mu F_{\alpha\beta}\right) \tag{9}$$

$$= -F^{\mu\alpha}J_\alpha + \frac{1}{\mu_0}g^{\mu\nu}F^{\alpha\beta}\left(\nabla_\alpha F_{\nu\beta} - \frac{1}{2}\nabla_\nu F_{\alpha\beta}\right) \tag{10}$$

where we have used the equation of motion $\delta S/\delta A_\sigma = 0$. The terms in parenthesis, antisymmetrized on α and β, equal

$$-\frac{1}{2}\left(\nabla_\alpha F_{\beta\nu} + \nabla_\beta F_{\nu\alpha} + \nabla_\nu F_{\alpha\beta}\right) \tag{11}$$

This vanishes because $F_{\nu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. So we're left with

$$\nabla_\nu T^{\mu\nu}_{EM} = -F^{\mu\alpha}J_\alpha \tag{12}$$

Dimensions

Let M, L, T, and I denote mass (in kilograms), length (in meters), time (in seconds) and current (in Amperes), respectively. The dimensions of the constants are

$$c \sim L/T \tag{13}$$
$$G \sim L^3/(M \cdot T^2) \tag{14}$$
$$\epsilon_0 \sim I^2 \cdot T^4/(M \cdot L^3) \tag{15}$$
$$\mu_0 \sim M \cdot L/(T^2 \cdot I^2) \tag{16}$$

Tensor components are coordinate dependent, so their dimensions depend on the dimensions of the coordinates. Let the coordinates have dimensions of length:

$$x^\mu \sim L \tag{17}$$

Then

$$g_{\mu\nu} \sim g^{\mu\nu} \sim 1 \tag{18}$$

and we can raise and lower indices without changing dimensions. Dimensions of other tensors:

$$R^\mu{}_{\alpha\nu\beta} \sim 1/L^2 \tag{19}$$
$$A_\mu \sim M \cdot L/(T^2 \cdot I) \tag{20}$$
$$F_{\mu\nu} \sim M/(T^2 \cdot I) \tag{21}$$
$$J^\mu \sim I/L^2 \tag{22}$$
$$T^{\mu\nu} \sim M/(T^2 \cdot L) \tag{23}$$

Electric and Magnetic Fields

The velocity of an observer is

$$U^\mu \equiv \frac{\partial x^\mu}{\partial \tau} \sim L/T \tag{24}$$

where $\tau \sim T$ is proper time along the worldline. It satisfies the normalization condition $U^\mu U_\mu = -c^2$. The electric and magnetic fields as seen by this observer are

$$E^\mu \equiv F^{\mu\nu} U_\nu \sim M \cdot L/(T^3 \cdot I) \tag{25}$$
$$B^\mu \equiv -\frac{1}{2c} \epsilon^{\mu\nu\sigma\rho} U_\nu F_{\sigma\rho} \sim M/(T^2 \cdot I) \tag{26}$$

The field strength tensor is

$$F^{\mu\nu} = \frac{1}{c^2} \left(U^\mu E^\nu - U^\nu E^\mu \right) + \frac{1}{c} \epsilon^{\mu\nu\sigma\rho} U_\sigma B_\rho \tag{27}$$

The scalar potential and electric charge density are:

$$\Phi \equiv A_\sigma U^\sigma \sim M \cdot L^2/(T^3 \cdot I) \tag{28}$$
$$\rho_e \equiv \frac{1}{c^2} J_\sigma U^\sigma \sim I \cdot T/L^3 \tag{29}$$

The 3-vector potential and the 3-current are defined by:

$$A^{(3)}_\mu \equiv \perp_\mu^\nu A_\nu \sim M \cdot L/(T^2 \cdot I) \tag{30}$$
$$J^{(3)}_\mu \equiv \perp_\mu^\nu J_\nu \sim I/L^2 \tag{31}$$

where

$$\perp_\mu^\nu \equiv \delta_\mu^\nu + U_\mu U^\nu / c^2 \tag{32}$$

is the spatial projection tensor for the observer.

Stress, Energy and Momentum

The stress–energy–momentum tensor is

$$T^{\mu\nu} = \frac{1}{2}\left(\perp^{\mu\nu} + U^\mu U^\nu/c^2\right)\left(\epsilon_0 E^2 + B^2/\mu_0\right) - \left(\epsilon_0 E^\mu E^\nu + B^\mu B^\nu/\mu_0\right) \tag{33}$$

$$- \frac{2\epsilon_0}{c} U^{[\mu}\epsilon^{\nu]\alpha\beta\gamma} U_\alpha E_\beta B_\gamma - (\Phi\rho_e - A^{(3)}_\sigma J^\sigma_{(3)}) g^{\mu\nu} \tag{34}$$

The energy density as seen by the observer is

$$(\text{energy density}) \equiv \frac{1}{c^2} T^{\mu\nu} U_\mu U_\nu = \frac{1}{2}(\epsilon_0 E^2 + B^2/\mu_0) + \rho_e \Phi \sim M/(L \cdot T^2) \tag{35}$$

The momentum density and energy flux,

$$(\text{momentum density})^\mu \equiv -\frac{1}{c^2} \perp^\mu_\alpha T^{\alpha\beta} U_\beta = \frac{\epsilon_0}{c} \epsilon^{\mu\nu\alpha\beta} U_\nu E_\alpha B_\beta \sim M/(L^2 \cdot T) \tag{36}$$

$$(\text{energy flux})^\mu \equiv - \perp^\mu_\alpha T^{\alpha\beta} U_\beta = \epsilon_0 c\, \epsilon^{\mu\nu\alpha\beta} U_\nu E_\alpha B_\beta \sim M/T^3 \tag{37}$$

are related by (energy flux) $= c^2$(momentum density). The momentum flux and spatial stress are defined by

$$(\text{momentum flux})^{\mu\nu} = (\text{spatial stress})^{\mu\nu} = \perp^\mu_\alpha T^{\alpha\beta} \perp^\nu_\beta \tag{38}$$

$$= \frac{1}{2}(\epsilon_0 E^2 + B^2/\mu_0) \perp^{\mu\nu} - (\epsilon_0 E^\mu E^\nu + B^\mu B^\nu/\mu_0) - (\Phi\rho_e - A^{(3)}_\sigma J^\sigma_{(3)}) \perp^{\mu\nu} \tag{39}$$

$$\sim M/(T^2 \cdot L) \tag{40}$$

Fermi Normal Coordinates

Let $\bar{x}^\mu$ denote Fermi normal coordinates (FNC) defined by the observer whose velocity is U^μ. Thus, $\bar{x}^0/c$ is the proper time along the observer's worldline. The observer defines a triad of vectors, e^μ_i that are Fermi-Walker transported along the worldline. These vectors are spatial, $e^\mu_i g_{\mu\nu} U^\nu = 0$, and orthogonal, $e^\mu_i g_{\mu\nu} e^\nu_j = \delta_{ij}$. The spatial coordinates of an event, $\bar{x}^i$, are the coefficients in the expansion of the vector $\bar{x}^i e^\mu_i$ that is tangent to the spacelike geodesic that connects the worldline to the event, and has magnitude equal to the length of the geodesic.

In Fermi normal coordinates, the metric on the observer's wordline is $\bar{g}_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and

$$\bar{U}^\mu = (c, 0, 0, 0) \tag{41}$$

$$\bar{e}^\mu_1 = (0, 1, 0, 0) \tag{42}$$

$$\bar{e}^\mu_2 = (0, 0, 1, 0) \tag{43}$$

$$\bar{e}^\mu_3 = (0, 0, 0, 1) \tag{44}$$

so that $\bar{U}_\mu = (-c, 0, 0, 0)$. We have

$$\bar{E}^i = \bar{F}^{0i} \tag{45}$$

$$\bar{B}^i = \frac{1}{2}\bar{\epsilon}^{ijk}\bar{F}_{jk} \tag{46}$$

$$\bar{\Phi} = c\bar{A}_0 \tag{47}$$

$$\bar{\rho}_e = \bar{J}_0/c \tag{48}$$

$$\bar{A}^{(3)}_i = \bar{A}_i \tag{49}$$

$$\bar{J}^{(3)}_i = \bar{J}_i \tag{50}$$

where the 3D Levi–Civita tensor is defined by $\epsilon_{\nu\alpha\beta} \equiv U^\mu \epsilon_{\mu\nu\alpha\beta}/c$. Thus, in FNC, $\epsilon^{ijk} = \pm 1$ if i, j, k is an even/odd permutation of 1,2,3. The components of the stress-energy-momentum tensor are:

$$(\text{energy density}) = \bar{T}^{00} \tag{51}$$

$$(\text{momentum density})^i = \frac{1}{c}\bar{T}^{i0} \tag{52}$$

$$(\text{energy flux})^i = c\bar{T}^{i0} \tag{53}$$

$$(\text{momentum flux})^{ij} = (\text{spatial stress})^{ij} = \bar{T}^{ij} \tag{54}$$